Analysis of a Propeller in Compressible, Steady Flow

Valana L. Wells*
Arizona State University, Tempe, Arizona

Abstract

THIS Synoptic describes an analytical method for solving the equation governing the inviscid, irrotational, compressible, potential flow about a propeller. The equation and boundary conditions are transferred to a noninertial system of coordinates rotating with the propeller, in which the basic problem becomes a steady one. The solution method takes advantage of the linearity of the model by superposing a "compressible" solution to the potential equation on an "incompressible" wake solution. In addition, the boundary conditions are satisfied by dividing the flowfield at the propeller plane, solving the equations separately ahead of and behind this plane, and enforcing continuity matching conditions. Applying the final boundary condition yields an infinite-series integral equation for the unknown circulation distribution. A lifting-line method is used to produce numerical results. Presented results establish the effect of compressibility on the induced field.

Contents

Except in the vicinity of the propeller and its wake, the flowfield caused by an advancing propeller in an inertial coordinate system may be considered irrotational and isentropic. This approach treats the viscous drag on the blade as inconsequential to the general induction theory. Consider a transformation of the linearized convective wave equation to a blade-fixed coordinate system that rotates with the propeller at angular velocity ω. This noninertial, cylindrical system (r, θ, z, t) is defined by $r=\tilde{r}$, $\theta=\tilde{\theta}+\omega\tilde{t}$, $z=\tilde{z}$, and $t=\tilde{t}$. Tildes represent quantities in the inertial, translating coordinate system. Performing these coordinate transformations gives the unsteady, linearized potential equation in the rotating system.

Although the general methodology works for unsteady flows, this Synoptic focuses on the aerodynamics of the steady propeller, that is, a rigid propeller experiencing a uniform freestream. This allows the time-dependent terms to be excluded, since the flow now appears steady in the rotating, translating coordinates. Converting the dimensional spatial variables to dimensionless form ($\rho=\omega r/U$ and $\bar{z}=\omega z/U$) and dropping the bar over the dimensionless z variable, the steady, potential equation becomes

$$\frac{(\rho\varphi_\rho)_\rho}{\rho}+\beta^2\varphi_{zz}+\frac{1-M^2\rho^2}{\rho^2}\varphi_{\theta\theta}-2M^2\varphi_{\theta z}=0 \qquad (1)$$

with $\beta^2=1-M^2$ and $M=(u/a)$ is the advancing Mach number.

This linear analysis assumes that all motion of the helicoidal wake occurs in the axial direction and that no distortion of the helicoid occurs. Although these assumptions have little basis in the actual behavior of the wake, they do have analogies in the linearized wing problem. The helicoidal wake approximation may be used for the linearized propeller regardless of whether it has the "Golsteinian"[1] circulation distribution that would produce exactly that wake shape. Because the helicoidal wake is retained, one can introduce helical or "Goldsteinian" coordinates following the wake contours. The new coordinate system is described by $\sigma=\theta+z$, which points along a helix, and $\zeta=\theta-z$. Note that $\zeta=$const describes a helix.
Goldstein shows that, in the far wake, the flowfield depends only on the variables ρ and ζ. This results from purely geometrical considerations and therefore holds for both compressible and incompressible flows. This result is of interest on two accounts. First, the perturbed flow in the far wake has an incompressible character, which implies that quantities that depend only on the far wake are invariant from incompressible to compressible flow. In addition, the far-wake solution for a propeller in incompressible flow also satisfies the compressible equation. The first of these points is significant. From energy considerations, it can be deduced that the propeller of minimum induced power will produce a wake that moves with constant rearward velocity at infinity. This condition holds for both compressible and incompressible flows. Therefore, since the compressible contributions to the induced field die out at some distance behind the propeller disk, the circulation distribution resulting in minimum power loss must be the same as that computed by Goldstein for a propeller in incompressible flow.

Formulating the problem in a rotating coordinate system leads to some mathematical consequences of interest. Because the freestream velocity depends on radial position, that velocity reaches the speed of sound at a certain sonic radius ρ_s. For a purely subsonic propeller, the tip radius ρ_a is always less than ρ_s. However, a solution to Eq. (1) must take into consideration the outer, supersonic region as well as the subsonic one in all cases. Therefore, the flowfield in the vicinity of a propeller will have both elliptic and hyperbolic character, regardless of the Mach number of the propeller itself.

As discussed previously, the solution to the equation for the incompressible velocity potential, $\varphi_i(\rho,\theta-z)$, also solves the compressible equation. Although it cannot capture the compressible or the three-dimensional effects at the propeller blades, this solution accurately describes the far-wake flow. Thus, a superposition of a compressible solution φ_c, which accounts for effects near the blades but dies out in the far wake, on the incompressible solution also constitutes a solution to the linear differential eqation. For the superposed form, $\varphi=\varphi_i+\varphi_c$, to be a solution to the entire problem, it must satisfy the problematic no-flow boundary conditions at the blade surfaces and the wake. (The conditions at $\rho=0$ and $\rho\to\infty$ are easily satisfied independently by φ_i and φ_c.) To account for these conditions, the field can be artificially divided into two regions—one forward of the propeller plane in which only the compressible velocity potential holds and the other behind the propeller plane in which both φ_i and φ_c apply. The form of φ_i, which automatically provides for the existence of a trailing wake, necessitates the division of the flowfield. However, this division requires the introduction of matching

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*Assistant Professor, Department of Mechanical and Aerospace Engineering. Member AIAA.

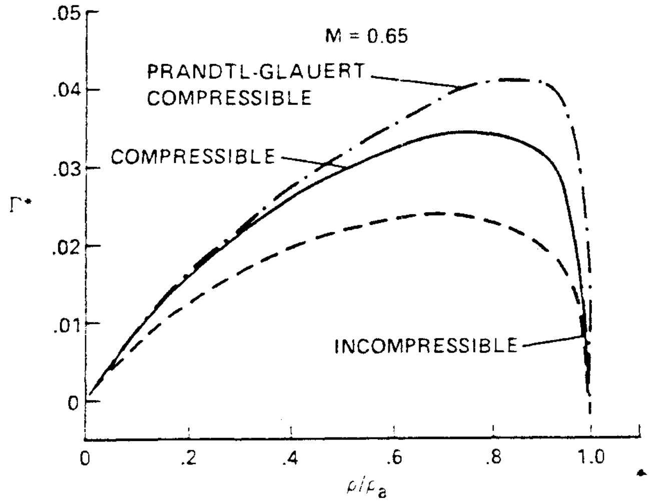

Fig. 1 Effect of compressible induction theory and comparison with Prandtl-Glauert correction.

conditions (continuous potential and continuous derivative) at the propeller disk, $z=0$, to ensure continuity of the flowfield. Davidson[2] first used this idea of superposition to solve for the flow around a propeller with specified loading in a wind tunnel. The current study develops the solution for free-air boundary conditions and arbitrary circulation distribution.

Reissner[3] proposed a potential solution to the incompressible propeller problem that, when integrated by parts, has a simpler form than that given by Goldstein:

$$\varphi_i = -\frac{B}{2\pi}\Gamma(\rho)(\theta - z) - \sum_{n=1}^{\infty} R_n \sin n\, B(\theta - z) \tag{2a}$$

where

$$R_n = \frac{(-1)^n}{n\pi}\left[K_{nB}(nB\rho)\int_0^{\rho} I'_{nB}(nB\xi)\xi\frac{\mathrm{d}\Gamma}{\mathrm{d}\xi}\mathrm{d}\xi \right.$$
$$\left. + I_{nB}(nB\rho)\int_{\rho}^{\rho_a} K'_{nB}(nB\xi)\xi\frac{\mathrm{d}\Gamma}{\mathrm{d}\xi}\mathrm{d}\xi\right] \tag{2b}$$

in which K and I represent the modified Bessel functions, Γ the circulation over the blade, ρ_a the dimensionless tip radius, and B the number of blades. Primes represent differentiation with respect to ξ. Some manipulation was required to obtain Eq. (2b) as the finite part of Reissner's original solution.

Because of the periodicity in the angular coordinate, the expected solution to the complete partial differential equation, Eq. (1), has the form

$$\varphi_c = \sum_{n=0}^{\infty} \varphi_n(\rho,z)e^{inB\theta} \tag{3}$$

where the real part of this sum is implied. Substituting this into the partial differential equation and performing a Hankel transformation on the equation gives a second-order, ordinary differential equation for the Hankel-transformed potential. Solving the resulting equation, inverting, and substituting into Eq. (3) gives

$$\varphi_c(\rho,\theta,z) = \sum_{n=0}^{\infty} e^{inB\theta}\int_0^{\infty} An(\gamma)\gamma J_{nB}(\gamma\rho)\exp\left[i\left(\frac{nB}{\rho_s^2}\right.\right.$$
$$\left.\left.\pm\frac{1}{\beta}\sqrt{\gamma^2 - \frac{n^2B^2}{\rho_s^2}}\right)z\right]\mathrm{d}\gamma \tag{4}$$

In the above expression for φ, the plus sign refers to the solution ahead of the propeller ($z \leq 0$) and the minus sign to the solution behind ($z \geq 0$); γ is the Hankel transform parameter that appears above as a dummy integration variable. It is evident that the character of solution in the z variable depends on the relative values of γ and nB/ρ_s because of the square root in the exponent. Since γ takes on the range of values from 0 to ∞ over the integration, a mixed flow characterized by both a decaying and a pure sinusoid appears for any nonzero, finite ρ_s.

It remains to determine the unknown coefficients $A_n(\gamma)$. This is accomplished through use of the continuity conditions at the propeller disk. At this point, the solution is complete except for specification of the circulation distribution $\Gamma(\rho)$. Resolution of the remaining unknown depends on the model used for satisfying the flow condition at the propeller blade. For the sake of simplicity, the following results were obtained through use of a lifting-line boundary condition at the propeller bound vortex locations, but the form of φ is completely appropriate for use in a lifting-surface model.

Figure 1, an illustration of the main result of this Synoptic, compares the compressible and incompressible circulation distributions for propellers with the same geometric angle-of-attack distribution. As expected, the compressible case has a greater magnitude of circulation, paralleling the same trend caused by the Mach number that occurs for fixed wings. Figure 1 also illustrates the error incurred by simply applying a two-dimensional Prandtl-Glauert, correction to an incompressible solution, which correction tends to overpredict the compressible circulation, a consequence of the planar geometry assumptions of the Prandtl-Glauert formula.

Conclusions

The development describes a direct method for solving the linearized potential equation for propellers in subsonic, steady flow. The following summarize the important conclusions of the foregoing work:

1) An analytic solution for the propeller was developed in compressible, steady potential flow as a function of circulation distribution.

2) The solution was implemented in a lifting-line formulation for use as an analysis tool. Results show that using a Prandtl-Glauert correction to an incompressible theory overpredicts the compressible circulation.

3) The minimum induced power propeller in compressible flow will have the same circulation distribution as the most efficient propeller in incompressible flow. Conditions producing that circulation will differ between the two cases.

4) Although not explicitly considered here, the solution should be valid for purely supersonic propellers since the development requires only that the disturbances be small relative to the freestream. The character of solution would be purely hyperbolic for this case, and the only singularity in the solution would appear if the freestream Mach number were 1. A mixed flow (subsonic-supersonic) propeller, however, would invalidate the linearization step because of the region of transonic flow.

References

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